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Physical bases of ballistic table computations

United States. Army. Ordnance Dept

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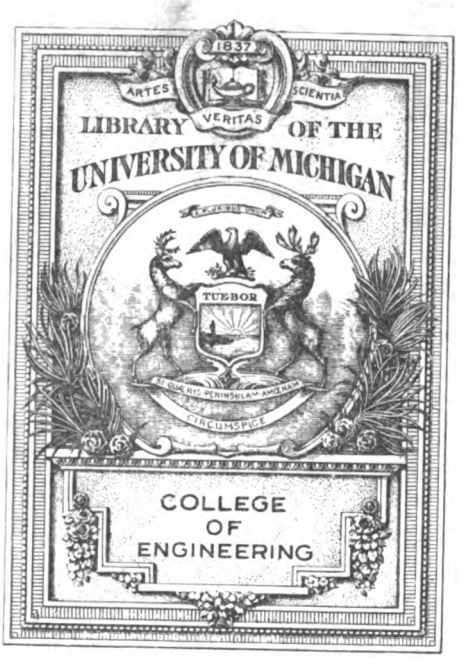

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PHYSICAL BASES

OF

BALLISTIC TABLE COMPUTATIONS

ORDNANCE TEXTBOOK

PREPARED BY
THE ORDNANCE DEPARTMENT

September, 1919

WASHINGTON
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1920

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WAR DEPARTMENT,
WASHINGTON, *September 30, 1919.*

The following publication, entitled "Physical Bases of Ballistic Table Computations," is published for the information and guidance of all concerned.

[062.1, A. G. O.]

BY ORDER OF THE SECRETARY OF WAR:

PEYTON C. MARCH,
General, Chief of Staff.

OFFICIAL:
P. C. HARRIS,
The Adjutant General.

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PHYSICAL BASES OF BALLISTIC TABLE COMPUTATIONS

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* This monograph constitutes Part I of the Introduction of New Ballistic Tables being prepared by the Ordnance Department.

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PHYSICAL BASES

OF

BALLISTIC TABLE COMPUTATIONS.

Prefatory Remarks.

These new tables for exterior ballistics were computed and compiled by the Ordnance Department and were designed and supervised by the author of this introduction. The circumstances demanding their construction will not be reviewed in this part, nor will any account be given here of their form and content, nor of the technique and special devices used in their computation. It is only the physical facts and theories upon which these tables are founded that will here be treated, and even the history of these theories will be left with little more than mention.

The matter treated here is qualitative rather than quantitative. The methods of numerical integration used in computing the trajectories, while in themselves only methods of approximation, are capable of giving results with any preassigned degree of accuracy, and as used yield vastly more precise figures than justified by the physical data available or the physical assumptions employed. This precision is obtained, however, at no extravagant sacrifice of labor, and secures results which are conveniently regular. Less accurate methods hitherto in vogue are now insufficient. The total number of physical factors in the problems of exterior ballistics is practically infinite. The elimination of all but a few is justified only by careful quantitative experiments and computations which are not discussed here. To catalogue and describe, merely, the various influences affecting the projectile in flight, leaves the relative importance assigned to each apparently haphazard and arbitrary. Thus this introduction seeks to mention only those physical assumptions which enter explicitly into the computations of the accompanying tables, and attempts only such justification for them as is in keeping with the expository nature of this paper.

The obvious phenomena of a trajectory can only be adequately treated by making use of the fact that the projectile in flight is a rigid body endowed with rapid rotation and undergoing precession and nutation as it traverses a path of double curvature, the motion for given initial conditions being determined by the continuous action of various forces, including the resistance of the air and gravitation. The old-fashioned spherical projectile presented a much simpler problem. When the accidental rotation was negligible, such features as drift, kiting, and coasting could not occur, and yaw was in the nature of events impossible. Dispersion with spherical projectiles is due chiefly to irregularities in jump of the gun, and in muzzle velocity, it being assumed that wind and atmospheric conditions are approximately unvaried in the interval of firing. With modern elongated projectiles the varying presentation of the axis of the shell to the air resistance is undoubtedly a significant factor in the magnitude of the dispersion. Except for an empirical formula for drift, the difference between the actual projectiles in use to-day and the spherical shot of former centuries has been entirely ignored in the quantitative theory of exterior ballistics. Even the dynamical constants of the shells, such as the position of the center of mass and the longitudinal and transverse moments of inertia, have not been measured for current ammunition (August, 1919).

This traditional attitude is much to be deplored, but on account of the present state of ignorance, and particularly for the sake of simplicity, the tables here given are computed as for spherical shells. In short, the shape, weight, and all other factors of the shell itself are regarded as effective through a single constant, the ballistic coefficient. Thus it is assumed that no matter what the actual shell may be it is always possible to determine a weight for a spherical shell of a preassigned diameter, such that the exterior ballistics of the two may be identical,[1] except for the one fact of drift, which quantity furthermore is not tabulated in this work.

I.—Air Resistance and Gravitation.

The most obvious influences affecting a projectile in flight are the two forces, gravitational force and resistance of the air. For convenience in computation these forces are replaced by proportional factors not involving the mass of the projectile, namely, the acceleration g due to gravitation, and the retardation, or

[1] Save in so far as the ballistic coefficient varies with the angle of departure and with muzzle velocity.

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counter-acceleration R, due to the air resistance. While for dense objects projected at low velocity, such as a stone thrown by hand, the effects of R are negligible in comparison with those of g, for light objects or for high velocities the relation is almost reversed, as is seen in leaves blown by the wind, and in heavy artillery, and, best of all, in the kite or aeroplane.

The Retardation R.

It is assumed that the retardation R can be written in the form $R = F\delta/K$, where F is a function of the velocity alone, δ is the air density, and K is determined by the weight and shape of the shell alone. As a first approximation this law has proved very satisfactory, although critical objections of many kinds may be urged. Recent experiments seem to establish the fact that F depends upon the form of the projectile as well as upon its velocity. From both theoretical and experimental investigations, F is a function also of the temperature of the air as affecting the velocity of wave propagation. This dependence upon temperature is taken account of in a correction term forming a supplementary table, but is ignored in the fundamental computations.

That R is proportional to the density of the air, δ, has never been contested, but, on the other hand, it has never been demonstrated experimentally. That K is constant along the trajectory—that is, that at various points in its flight the retardation of the projectile by the air is a function only of velocity and air density—is not credible for the elongated projectiles of to-day, but is nevertheless a satisfactory approximation for present purposes.

The Air Resistance Function $F(v)$.

The function $F(v)$ is also written as $vG(v)$ or as $v^2B(v)$, the occasion for these variant forms being that $F(v)/v = G(v)$ appears explicitly in the computation used in these tables, and $F(v)/v^2 = B(v)$ is of theoretical interest, since it is limited and for low velocities or for rough estimation may be approximated by a constant. With the so-called "square law" or "quadratic law," B is regarded as constant, so that $F(v)$ varies as v^2. The function $B(v)$ occurs explicitly in several methods of computation.

The G function used in the computation of these trajectories consists of but part of the table used for general computations in the Ordnance Department. This part itself consists of two portions: one for v extending from 0 to 600 meters per second, the second for v extending from $v = 600$ to 1,200 meters per second. The G tables at hand extend, however, up to about 1,800 meters, this latter portion being obtained by a method of extrapolation. Both portions here used are taken from French tables based on the results of the Commission de Gâvre, as found, e. g., in Table V of Charbonnier: Balistique Extérieure. The second portion was taken without modification from Charbonnier, save for the introduction of the factor v where required, and a constant numerical factor. The first and more critical part differs very slightly from Charbonnier's tables in several places due to careful smoothing out on a large scale. This smoothing out was accepted as given in the French pamphlet, "Commission de Gâvre: Calculs des trajectoires par arcs, Lorient Imprimerie des Constructions Navales, September, 1917." From this pamphlet we translate the following prefatory remarks concerning $F(v)$ and $B(v)$, where $F(v) = vG(v) = v^2B(v)$, as functions of v. "This function [$F(v)$] was obtained by regularizing the experimental results which constitute the subject matter of reports Nos. 1414 and 1429 of the Gâvre Commission (Mar. 14 and July 21, 1888)." * * * "It has seemed useful therefore to replace the function $F(v)$, or rather $B(v)$, which is used directly in the computation by small arcs (as tabulated previously), by a function presenting a more continuous character. For this purpose we chose at first the following formula due to Capt. Demogue, of the Metropolitan Artillery:

$$B = 0.2550 + \frac{\sqrt[4]{1 + 0.0392\left(\frac{v'}{10}\right)^2}}{27266 + 494v'^2} \arctan v',$$

where $v' = \frac{v - 330}{50}$ and arctan v' is expressed in sexagesimal minutes * * *. An even closer correspondence was, however, sought, and after having calculated the discrepancies at intervals of 10 meters for v (and in the region $200 < v < 460$, at intervals of 5 meters), the following compensating formula was finally adopted:

$$\log B = \log\left[0.2550 + \frac{\sqrt[4]{1 + 0.0392\left(\frac{v'}{10}\right)^2}}{27226 + 494v'^2} \arctan v'\right] + \frac{v^{\mathrm{H}} - 6}{10},$$

where v^{H} means v measured in hectometers.

"It is this function, then, calculated under direction of Chief Engineer Garnier, that figures in the present pamphlet. *It makes no claim to better represent the laws of air resistance than the tables in current use, but to agree as well as possible with them while having greater continuity than they have.*"

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The exact relation between the French tabulated functions $F(v)$ and $B(v)$ to $G(v)$ is given by

$$\begin{cases} 10+\log G=\log B+\log v+7.0570. \\ 10+\log G=\log F-\log v+7.0570. \end{cases}$$

It is to be hoped that within a few years experimental improvements in shell design shall have so reduced the resistance that, for this reason if for no other, the resistance laws discovered in 1888 will be obsolete and must be discarded. In such an event the present tables must be replaced by others computed afresh on the basis of these revised functions.

The Ratio δ/K.

Since the quantities δ and K occur only in the form of the ratio δ/K, it is clearly immaterial whether or not a common factor be viewed as introduced into both quantities. The air density over a given restricted area is usually treated as a function of altitude alone and supposed to differ little from a standard form. In place of δ it is usually convenient to take $H(y)$, a standarized and tabulated function giving the ratio of the density at altitude y to the density at altitude zero. In other words, for δ one writes $\delta_0 H(y)$, where δ_0 is the density at altitude zero. The value of δ_0 is divided into K to give what is written as C and is also called the *ballistic coefficient*. Thus, while K is dependent only on the shell, the value of C depends upon the density of the air at the points of altitude zero. We therefore write $\delta/K=H(y)/C$. Variations from the standard values of $H(y)$ are taken care of in supplementary tables.

Now, the function $H(y)$ is taken as an exponential function of the form $H(y)=e^{-.0001036y}=10^{-.000045y}$, where e is the base of the Napierian logarithms, y is measured in meters and the algebraic sign of y is taken positive as y is measured upward. Now the factor $1/C$ can itself be expressed as e^{-hy_1}, where h is the quantity .0001036 and y_1 is a sort of pseudo altitude. Thus, $H(y)/C$ may be written $e^{-h(y+y_1)}$ and all effects due to the shape and structure of the projectile and density of the air at altitude zero are expressible by the use of a pseudo altitude y_1 added to the actual altitude and constant throughout the trajectory. In view of this simple character of the H function, the level of zero altitude may be taken at any preassigned height desired, at sea level, at the muzzle of the gun, or at some point high in the air, the only difference being a corresponding modification in the reference point for y_1.

While the function $B(v)$ is frankly a conventional average of a large number of obsolescent data, its standing in ballistic theory is widely recognized, and until further facts are learned its continued use in essentially the same form by all nations is to be expected. The form of the air density function $H(y)$ rests on data much more satisfactorily known experimentally and theoretically, but yet is assumed differently in different nations. For this reason, a few words will be said for the usage here adopted.

The Density Function.

The density of the air as a function of the altitude in common with other meteorological phenomena appears to be incapable of adequate expression in simple terms. Weather conditions exhibit in marked degree features of dynamical instability that have been classed under the term "turbulence." The density of the air at a given altitude over a given place shows large variations of a strictly transient character incident to the changing geographic distribution of centers of high and low pressure. The best that can probably be done for a simple constant law is to seek an average over a large number of observations under conditions restricted only with respect to explicitly selected variables. As a result of such averaging, notable differences are still observable according to seasons, according to time of day, according to geographical locality, and other features. A few charts and tables are available showing monthly, seasonal, and annual averages at various places, in terms of the altitude, and may be obtained from the United States Weather Bureau. Some impression of the exceeding complexity of the subject may be obtained from consulting the subject "Meteorology" in the Encyclopedia Britannica.

The following remarks are of a general nature. The atmosphere has been divided roughly into two layers, the boundary between the two being the limit of convection from the earth's surface. The lower, known as the troposphere, extends in moderate latitude up to 9 kilometers and then merges, sometimes abruptly, into the upper, or stratosphere, which extends from 12 kilometers upward to an unknown height, but not less than 40 kilometers. In the tropics the merging may occur at about 16 kilometers. The stratosphere is characterized by practically isothermal conditions vertically, although of course the temperature varies steadily with latitude. The stratosphere is also relatively undisturbed by cyclonic pressure variations and rainfall. This region lies outside of these tables, which are restricted to about 10 kilometers in altitude.

The density of the air at any given altitude is obviously dependent upon the barometric pressure at that point, which is in turn largely determined by the integrated density of the superincumbent layers. Under the

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simplest hydrostatic assumptions and the use of Boyle's law, it is possible to derive the pressure and density of the air as functions of the altitude from an assumed variation of the temperature as a function of the altitude. However, the average conditions are but roughly in accordance with the adiabatic law of the change of temperature with pure gases, and the equilibrium, if existent, is certainly dynamical rather than statical in character. This is apparently due not so much to the different constitution of the air with increase in altitude as to the distribution of dust particles and water vapor, with resulting changes in radiation, the persistence of convection currents, and the interaction of diurnal and latitudinal variations. A very convenient feature, which is apparently true, is that variations in barometric reading due to cyclonic areas at high and low pressure passing over a given spot give an increase or decrease noticeable, with negligible variations with altitude, throughout the troposphere, i. e., at all altitudes up to about 11 kilometers at moderate latitudes. This is known as "Egnell's Law."

Within the troposphere, above about half a kilometer from the ground, a fair average regularity may be noted. A very satisfactory representation is the exponential one used in these tables, according to which the ratio of the density at altitude y to the density at altitude $\bar{y}$ (these being in meters), is $e^{-.0001036(y-\bar{y})}$ or $10^{-.000045(y-\bar{y})}$. A seasonal correction might be introduced for greater exactness. Unfortunately, this correction can not be treated as a mere factor, since at about 8½ kilometers altitude the average seasonal density remains the same for different seasons. The correction must be made in the exponent h, and depends upon the geographical character of the place considered and upon the time of year. The results here given are intended to apply only to air aloft. The variation of air density with altitude up a mountain side follows a different law, usually assumed to be a fall of .1 of an inch in mercury per 100 feet.

Within the first half kilometer in altitude at any one point the diurnal density changes as affected by temperature are very marked, but are hardly noticeable above this height. This fact makes measurements at the ground of little value for long trajectories unless supplemented by data taken at least 500 meters above the ground. The practice of extrapolation to sea level gives data taken at inland points an occasionally misleading appearance.

It can not of course be expected that any direct averaging of observed results will exactly coincide with an exponential law. Certain consistent sources of discrepancy are obvious. To mention one: more observations have been taken in summer and also in good weather; that is, under conditions of relatively high barometric pressure. To assume the law as an exponential one results in more than merely a smooth curve with convenient analytic form. If the law be exponential, the density function may be combined with the ballistic coefficient so that the effect of a change in the ballistic coefficient may be interpreted as equivalent to a mere change in altitude. By use of this fact the independent parameters of a trajectory are reduced from four to three, and the construction of ballistic tables reduced from one of immensely laborious magnitude to practicable proportions. The corrections to be made at the time of firing for variations in density from any assumed law are never very great for the trajectories here considered, and yet they are but seldom entirely negligible. The advantages of replacing an exponential density law by one more immediately the result of averages are questionable; the disadvantages are overwhelming.

Entirely different conditions hold in the stratosphere, where an exponential law is well established on theoretical and observational bases. This exponential law is different, however, from that in the troposphere, in that the constants have different values in the two cases.

If an exponential density law $H(y)=e^{-hy}$ be assumed, y being measured upward, the value of h, chosen to agree most closely with observations, shows a seasonal variation. The mean value of h, as a result of observations covering several years, is given by the French (Cours de Balistique—G. Sugot, 1918), as follows:

	Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Mean value of $10^4 \times h$	1.14	1.12	1.09	1.05	1.02	1.00	0.98	0.97	0.99	1.08	1.11	1.12
Mean error	.05	.03	.00	.015	.03	.01	.03	.015	.00	.06	.08	.06

M. Brazier, meteorologist of the Central Meteorological Bureau of France, after having shown that the h of the formula $H(y)=e^{-hy}$ may be expressed as

$$h=k\frac{\frac{p_0}{T_0}-\frac{p_y}{T_y}}{p_0-p_y},$$

where k is a physical constant, p_y is the pressure, and T_y the absolute temperature at the altitude y, has calculated *a priori* the mean values of h at altitude 3 kilometers for the various months, making use of the data

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collected by the meteorologists, viz, the monthly distribution of barometric pressure and of temperature, and the monthly differences in temperature from that at the earth to that at the altitude 3 kilometers observed at Lindenberg. He then found the following for Lindenberg:

	Jan.	Feb.	Mar.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
$10^4 \times h$	1.12	1.08	1.07	1.04	1.00	0.98	0.98	0.98	1.02	1.06	1.09	1.11

On this basis 1.036 is a fair practical mean for $10^4 \times h$, since firings in December and January may be expected to be less common than in the summer. Of course if yards rather than meters be used as fundamental, the advantage of $10^4 \times h = 1.036$, which is that $H(y) = 10^{-.000045y}$, is largely lost and the nearest *two-figure* expression with ten as a base would be used.

The Gravitational[1] Acceleration, g.

Gravity is a term for the attraction between material bodies. The Newtonian law of universal gravitation is that every mass attracts every other mass with a force which varies directly as the product of the attracting masses and inversely as the square of the distance between them. The law implies in particular that the force exerted is independent of the temperature or velocity of the moving body. Both of these conclusions have been attacked of late years, but it is not questioned that they are sufficiently exact for ballistic purposes.

It is easily shown mathematically that a sphere whose density varies only with the distance from the center attracts an external body exactly as if the whole mass were collected at the center, and that a similarly constituted spherical shell—i. e., a mass bounded by two concentric spherical surfaces—while attracting an external body as though its mass were collected at the center, exerts no attraction at any internal point. Let us apply this to a point in the atmosphere at height y above the ground, regarding the earth as a perfect sphere of mass M and of radius R, and assuming the density, whether of the earth or of the atmosphere, to vary only with the radial distance. The atmosphere outside the spherical surface of radius $R+y$ exerts no attraction, while the atmosphere within this surface, of mass denoted by M', attracts as if collected at the center. Thus the total attractive force at the point upon an attracted body of unit mass is $G(M+M')/(R+y)^2$, where G is a constant. This becomes $g_0(1+M'/M)(1+y/R)^{-2}$, where $g_0 = GM/R^2$, is the corresponding force at the earth's surface, i. e., wholly within the atmosphere. Counted in kilograms, M' is large, but even if we went to the supposed confines of the atmosphere M'/M would be less than one millionth. Thus for ballistic purposes the attraction of the atmosphere may be neglected. The variation with height in the earth's own attraction is much more important. At all heights attained by projectiles the second order term, i. e., in y^2/R^2 and those of higher orders obtained in expanding $(1+y/R)^{-2}$, may be neglected, giving for this expansion merely $1-2y/R$. At a height of 10 kilometers this represents a reduction of 1 part in 320 in gravity.

In reality the earth is not a sphere, but approaches to a spheroid whose equatorial radius is 21.6 kilometers longer than its polar radius. Also it rotates and the "centrifugal force" due to rotation reduces gravity, especially near the equator. The earth's surface is also irregular in outline and the density variable, at least near the surface. Thus the formulæ actually advanced to show the variation in gravity at different parts of the earth's surface are complicated.

The following formula due to Helmert is perhaps the best known. In it g denotes the acceleration of gravity in C. G. S. units, i. e., in cm/s^2, whereas this must be divided by 100 to obtain the figure in m/s^2 to correspond to that of the tables.

$$g = 978.000(1+0.00531\sin^2\Phi) \times \left(1 - \frac{2y}{R} + \frac{3}{2}\frac{y}{R}\frac{\delta}{\Delta} - \frac{h'(\delta-\theta)}{2R\Delta} + q\right).$$

Here Φ is the latitude, R the earth's mean radius, y the height above sea level, h' the thickness of surface strata of low density, Δ the earth's mean density (approximately 5.6 times that of water), δ the mean density of surface strata (usually taken as 2.8), θ the actual density of surface strata for the region, and q a so-called orographical correction arising from neighboring mountains, and so on. At sea level, supposing q negligible and $\theta = \delta$, this becomes

$$g = 978.000(1+0.00531\sin^2\Phi),$$

or more conveniently

$$g = 980.5966 - 2.5966\cos 2\Phi.$$

Thus g has its mean value where $\cos 2\Phi$ vanishes, i. e., where $\Phi = 45°$, which explains why it is usual to

[1] In substance, an extract from the Meteorological Glossary (British Meteorological Office, 1918).

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reduce gravity to latitude 45°. The value of g at Kew Observatory is taken as 981.200. The formula gives g at the pole as 983.19, and at the equator as 978.00. At Greenwich it is 981.19.

Over any extended trajectory the curvature of the earth enters not only as affecting the height of site, since a horizontal plane through a point at sea level can not pass through a distant point at sea level, but also as producing a convergence in the lines of gravitational force. In a rectangular coordinate system as used in computing trajectories, the gravitational force is regarded as directed parallel to the y-axis. As the trajectory is extended, the direction in which gravity acts is approximately toward the earth's center which will not be parallel to the initial y-axis, and a "horizontal component" of gravity is introduced, "horizontal," that is, with respect to a rectangular system which at the initial point has vertical and horizontal axes.

For the restricted trajectories of this table these several variations in the gravitational theory are ignored. A constant value $g = 9.80\ m/s^2$ is used as the basis. In many of the trajectories a variation of 1 or 2 per cent in g would be negligible. This value is not only more convenient in computation, but seems theoretically as good as 9.81 often employed, in view of the moderate latitudes in which it may be expected that much of the firing in our service is to be conducted.

II.—Standard Conditions and Differential Corrections.

The data entered in range tables are computed with reference to certain standard conditions and are correct only under what are known as range table assumptions which, with rare exceptions, include the following:

1. The earth is motionless.
2. The gun and target are in the same altitude above sea level.
3. The preassigned standard muzzle velocity for that type of gun is actually obtained.
4. There is no wind.
5. The atmospheric density varies regularly with altitude according to the exponential law given by the H function, and is standard at the muzzle (15° C. = 59° F., 750 mm. = 1000 mb, 1.2034 kg/m³).
6. The action of gravity is uniform in intensity, is directed toward the earth's center, and is independent of the geographical location of the gun, $g = 9.80\ m/s^2$.
7. The function $F(v)$, which is a factor of the retardation R, is a function of the velocity v, alone, and has the tabulated values.
8. The factor C, which depends upon the weight, finish, figure, and internal mass distribution of the projectile, as well as upon air density, and which for standard air density at zero altitude reduces to K, is a constant on the trajectory and is as determined from experimental firing. This includes the assumption that all precessional and nutational effects of the projectile may be ignored, and hence the trajectory lies in a vertical plane.
9. The jump of the gun is as assumed from observations.

In certain cases range tables are constructed under assumptions which may differ slightly from these. Thus the curvature of the earth may be neglected, and the dependence of $F(v)$ upon the temperature may be utilized by reference to an assumed standard drop of temperature with altitude, and in extreme cases the non-uniformity of intensity of gravitational attraction may be taken into consideration.

In range tables the dependence of C upon the weight of the projectile may be explicitly analyzed, so that not only C as a whole but the weight of the projectile is assumed as specified. Similarly the muzzle velocity may be assumed to depend upon the temperature of the powder according to a stated formula.

What is meant by "fundamental *ballistic* table assumptions" is a much simpler set of conditions. For ballistic tables there is no particular projectile, gun or powder case, viewed as standard to the exclusion of others. The caliber of the gun, weight and figure of the projectile, angle of elevation, standard velocity, and other range-table constants never appear in ballistic table work. The only factors of the problem considered at all are actual velocity, angle of departure, ballistic coefficient C, and the meteorological and, more generally, geophysical forces which act along the path of the projectile.

The fundamental ballistic table assumptions are as follows:

1. The earth is motionless.
2. There is no wind.
3. The atmospheric density varies regularly with altitude according to the exponential law given by the H function and is standard at the muzzle.
4. The action of gravity is uniform in intensity, is directed toward the earth's center, and is independent of the geographical position of the gun, $g = 9.80\ m/s^2$.
5. The function $F(v)$, which is a factor of the retardation R, is a function of the velocity v, alone, and has the tabulated values.
6. The factor C is constant and known.

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Certain discrepancies between actual and assumed conditions may be so conspicuous as to demand corrections. These corrections are of two types, permanent and transitory. Of permanent corrections the following are tabulated:

1. Corrections in range and deflection for the rotation of the earth as functions of geographical latitude of the gun, azimuth of the plane of fire, and the three standard parameters of the trajectory, e. g., *M. V.*, ϕ, and *C*.

2. The corrections in range due to replacing a uniform temperature by an assumed standard temperature structure of the atmosphere in so far as the variations in temperature affect $F(v)$.

The following transitory corrections are also tabulated:

1. Corrections in range for variations from assumed standard in air density throughout layers at convenient altitude intervals.

2. Correction in range for a component of wind in the plane of fire throughout layers at convenient altitude intervals.

3. Correction in deflection for a component of wind at right angles to the plane of fire throughout layers at convenient altitude intervals.

4. Corrections in range for variations from assumed standard temperature structure as affecting $F(v)$ throughout layers at convenient altitude intervals.

These two sets of corrections are known as the standard differential corrections for ballistic tables. Since trajectories with maximum ordinate of more than 10 kilometers are not included within the scope of these tables, variations in g are ignored. Corrections due to atmospheric moisture, except perhaps when in the form of mist or rain, etc., are negligible.

For the use of these tables, the gun and target may differ in altitude by a very large amount. Thus the tables are adapted equally to antiaircraft fire, drop bombs, mountain firing, etc., as well as to firing on a level. The once familiar theory of the "rigidity of the trajectory" becomes for these purposes wholly obsolete. For firing on a level no supplementary correction table of the earth's curvature is needed save as an incident in constructing the original table.

The Rotation of the Earth.

For most commercial purposes the theory of a motionless earth is entirely satisfactory. Of course, sunrise and sunset would be explained by a diurnal motion of the sun about the earth, and for those who insist on an astronomic theory reference need merely be made to the Ptolemaic hypothesis. The Foucault pendulum, the gyroscopic compass, and the oblateness of the terrestrial spheroid suggest at once, however, that the earth should be regarded as rotating, and the explanations of the general circulation of the atmosphere are directly based on this fact. What must otherwise appear for long trajectories as discrepancies in observation are in part predictable consequences of this rotation of the earth.

There are three correction coefficients, A, B, D (the third is called D rather than C to avoid confusion with the ballistic coefficient). We shall consider separately, firings at the pole and at the equator, as illustrating limiting cases.

Suppose a gun at the equator is aimed due east. The rotation of the earth being symmetrical with respect to the equatorial plane, the earth's rotation will not of itself cause deflection for a spherical shell. The shell being given an initial velocity, it would proceed in a straight line with undiminished speed were it not for the resistance of the air and for gravitation. While for any quantitative estimate the resistance of the air must be taken into account, some qualitative notion of the corrections required may be obtained by neglecting the atmosphere. The projectile in vacuum would preserve its horizontal component of velocity unimpaired were the earth a fixed plane. The earth being neither fixed nor plane the term "horizontal" as applying to a component of velocity must be used with caution, since any object moving at a fixed altitude above sea level in the equatorial plane would move in a curved surface, not in a "horizontal plane."

If a gun be placed at the equator and be aimed in an east and west plane, the projectile will start out on its trajectory with a muzzle velocity V, and angle of departure ϕ, both relative to the earth. If the earth were not in rotation and there were no atmosphere any original uniform translation of the system would not in any way have affected the measured range. For a uniform translation would mean constant difference between the actual and relative horizontal components of motion of the projectile that would not affect the time of flight, maximum ordinate, nor range measured on the earth. When we still ignore the atmosphere but suppose the earth rotating rather than being translated, the situation is altered. Owing to the divergence of lines through a center of rotation, the distance traversed at higher altitudes to give the same range would be greater than at lower altitudes, while it is only linear velocity, not angular velocity about the earth's center, that is imparted to the projectile. Thus a projectile with a uniform "horizontal" velocity would appear to an observer on the earth's surface to be losing some of this while rising, and regain it while falling. But for the

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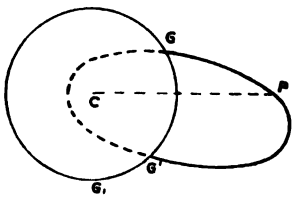
rotating earth there is no preservation of "horizontal" velocity, since a line at one point horizontal if continued becomes apparently oblique, owing to the curvature of the paths of points on the earth's surface.

For even the simplest quantitative discussion of the condition *in vacuo*, it is best to treat the projectile in the same manner as used in astronomy for a satellite. The Newtonian law of gravitation will therefore be assumed, and incidentally Kepler's law, that a satellite will move in a plane and so as to make the radius vector from the attracting mass to the satellite sweep out equal areas in equal times.



In the above figure let C represent the earth's center and let the circle shown with C as center represent the equatorial section of the earth. The position of the gun at the instant of firing as denoted by G and P is the position of the projectile at any time t. The path described by P is of course part of an ellipse under any initial velocity at present attainable in artillery fire, the completed figure containing a fictitious portion within the earth's surface and having C as a focus. The radius vector $\overline{CP}$ sweeps out equal areas in equal times. If the velocity of the projectile at the muzzle of the gun were in such a relation to the angular velocity of the earth's rotation that the radius vector $\overline{CP}$ at the instant when P was at G was turning at the same rate as the earth, this would only mean that to an observer on the earth the gun was aimed vertically upward or toward the zenith. Suppose this the case. When the projectile reaches the point G', where the elliptical trajectory reaches the earth again, the radius vector $\overline{CP}$ is once more moving with the velocity of the earth. At all intermediate points on the trajectory the radius vector has been moving more slowly, so that when the projectile reaches G' the gun will be in some such position as G_1. The projectile would appear to fall vertically to an observer at G', but to have suffered a displacement $\overline{G_1G'}$. There is therefore a correction term for fire in the equatorial plane which for a gun aimed toward the east or west becomes a range correction. This correction term is proportional to the angular velocity of the earth and to the diameter of the equatorial section.

For fire along a parallel of latitude, that is, in a section parallel to the equator, a situation obtains which, while not entirely similar, is analogous so far as terms of the first order of magnitude are involved. The section is of smaller diameter, merely, so that the correction term must contain the factor $\cos l$, where l is the geographical latitude of the gun.

For a gun on the equator and firing along a meridian the effect of the rotation of the earth on range is negligible, the above correction now becoming one of deflection. Thus the correction in range is proportional to $\sin a$, where a is the azimuth of the plane of fire measured from the south through the west. The correction term for range here discussed is indeed shown by mathematical analysis to be of the form

$$E \cos l \sin a,$$

where E contains the angular velocity of rotation of the earth as a fixed factor and is only computed by rather elaborate integrations from the data of the motion at all points along the trajectory.

This correction term would occur even if the earth were regarded as a cylinder. Indeed, if at a point of the earth's surface a cylinder of rotation be drawn coaxial with the earth, its radius would be proportional to $\cos l$ and would account for this factor, the factor $\sin a$ would enter from the same considerations as before.

In the same manner a decomposition of the correction whose total magnitude is $E \cos l$, gives not only $E \cos l \sin a$ for range correction, but the other component $E \cos l \cos a$ for deflection correction, deflection being measured in units of length as a displacement and not as an angle. The terms $E \cos l \sin a$ and $E \cos l \cos a$ are displacements upon the cylinder. Now, except at the equator, a cylinder is a very poor approximation to the earth's surface. To obtain the displacements on the tangent cone, further corrections containing the factor $\tan \omega$, where ω is the angle of fall, may be introduced. These differ for range and deflection. There are indeed two correction terms, viz: $-A \cos l \sin a$ for range and $D \cos l \cos a$ for deflection, where $-A = D - B \cot \omega +$ *terms of fifth and higher orders in the time of flight.* These terms of higher order account for the air resistance, etc.

There is, however, another deflection term arising in an entirely different manner. To a person on the equator there is no apparent right or left motion of the heavens along the horizon, while to an observer at a pole all low-lying objects of the celestial hemisphere appear to move horizontally only. A tangent to a meridian at any point appears with respect to distant stars to generate a cone of semivertical angle

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equal, numerically, to the latitude. For a point on the equator this cone reduces to a line, and at a pole the cone spreads out into a plane. A projectile fired from the pole would undergo an angular deflection approximately proportional to the time of flight, and hence a lateral displacement approximately proportional to the product of the time of flight by the range. The corresponding effect at latitude l is equal to the effect at the pole multiplied by $\sin l$. Thus there is a deflection term $B \sin l$, where a rough approximation of B is $\Omega X T$, Ω being the angular velocity of rotation of the earth, X the range, and T the time of flight.

In conclusion, it can be proved by mathematical analysis that except for negligible secondary effects, the range displacement ΔX and lateral displacement ΔZ occasioned by the rotation of the earth are respectively as follows:

$$\Delta X = -A \cos l \sin a$$
$$\Delta Z = B \sin l + D \cos l \cos a$$

where A, B, D are computated by integration, and depend upon the data of each given trajectory.

The actual effects upon the trajectory due to the rotation of the earth are only roughly approximated when the resistance of the air is left out of account, owing to the marked modifications in terminal velocity resulting from the continued operation along the trajectory of air resistance. For conditions in vacuum, the angle of departure ϕ is equal to the angle of fall ω. The values of A, B, D in this simple case may be expressed without integrations as follows, except for terms involving the consequences of the gravitational field, etc.:

$$A = \Omega X T (\cot \omega - \tfrac{1}{3} \tan \phi)$$
$$B = \Omega X T$$
$$D = \tfrac{1}{3} \Omega X T \tan \phi$$

whence $-A = D - B \cot \omega$ in this case.

It may be noted that in vacuum, A changes sign at $\cot \omega \ (= \cot \phi) = \frac{1}{3} \tan \phi$, or at $\phi = 60°$. Thus for trajectories in vacuum with angle of departure as 60°, the range correction is inappreciable at all latitudes and for all azimuths of the line of fire.

Method of Taking Account of Density Corrections.

Any variation throughout an altitude layer in the density of the air different from that supposed by the H function can enter into the computation only through the coefficient C. Thus to account for density variations, the quantity C is temporarily varied. If the data for a trajectory are given at all points, this variation is easily taken into account. When the original trajectory reaches the bottom of the layer in question, the elements of the trajectory at this point are selected. The C is then changed as needed and the trajectory continued to the upper surface of the layer. With the data at this point and the original C, the computation is continued again, or making use of the tables at each step, the results are merely read and applied.

The degree of humidity, as well as innumerable other variable factors, is not taken into account explicitly, except occasionally in the more complete statements of standard conditions.

The Effects of Wind.

The fundamental equations of motion of a projectile are usually obtained under simple assumptions, among others that there is no wind. The effects of the wind, while relatively slight, are not to be totally disregarded and these will be here investigated in order to secure correction terms to account for these effects.

In still air the motion of the projectile is determined from given initial conditions by the influences of air resistance and gravitation. Now, the resistance of the air is solely a phenomenon due to relative motion of the projectile in the surrounding medium. Were the air to be conceived of as accompanying the projectile in its flight, there would be no air resistance. Thus the magnitude of the air resistance is a function solely of the algebraic difference of velocities of air and projectile, other things being equal. It has been assumed that the factor in the air resistance which has been called the ballistic coefficient is constant on the trajectory. This depends on the form of the projectile as presented to the resisting medium. Any alteration of presentation, such as yaw, alters the center of pressure and the amount of resistance. If the motion of the projectile with reference to still air be supposed to be such that the axis of figure of the projectile is always tangential to the trajectory this situation may not persist with respect to the relative motion in a wind with nonvanishing transverse component. However, the velocity of the wind is in any case small in comparison to that of the projectile and the influence of the moving air is usually regarded, furthermore, as varying the axis of the shell by virtue of gyroscopic reaction, so that at most the value of C is changed in a negligible fashion.

An effect of the wind is in general to cause lateral displacements. In addition to the original horizontal and vertical axes taken in the plane of fire it is necessary to introduce a third coordinate perpendicular to

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the plane of fire, and it is convenient to use also a moving system of coordinates relatively stationary with respect to the wind. Setting up the differential equations of motion for this case, the simple but interesting result is obtained that the space curve described by the projectile will lie in a vertical plane when referred to axes moving with the wind. So far as the projection of the trajectory in the plane of fire is concerned, the equations with respect to the moving axes are identical in form with the fundamental equations, actual velocities being, however, replaced by velocities relative to the wind. For example, the range obtained with a uniform rear wind is that which is given with no wind in a trajectory whose initial horizontal component of velocity is the actual horizontal component minus the velocity of the wind, vertical components being unaltered, plus the distance traversed by a given particle of air during the flight of the projectile. This is sometimes called the theorem of the Relation to the Moving Atmosphere.

The Effects of Temperature Changes.

The temperature of the air enters into ballistic computations in two distinct ways as affecting (1) the density and (2) the elasticity of the air. For constant pressure a given quantity of air changes density with temperature approximately according to Boyle's law. The precision refinements of Van der Waal's corrections may well be ignored, but dynamical features are likely to modify the situation. Many kinds of temperature variations at the earth's surface are classified. Among the variations with time there are the superimposed changes in temperature of diurnal and seasonal character, and irregular variations depending upon less obvious meteorological factors. The geographical variations roughly determined by latitude and by the distances from continental confines are not less pronounced. The diurnal variations are damped out rapidly with elevation above the earth's surface, but seasonal and geographical variations are almost as pronounced in the upper strata as at the earth's surface. This is due in part to the persistence of horizontal and vertical convection currents of great importance resulting in an interchange of air over oceans and continents and between poles and equatorial regions, and is an incident of the fundamental phenomena of the so-called general circulation of the atmosphere. The effects of an irregular temperature variation may, however, be overemphasized as far as density is concerned. Pressure is of course higher in an anticyclone than in a cyclonic depression, and it has recently been made sure, at least for certain observation stations, that above 1 kilometer the air in the high pressure is warmer than in the low pressure, so that the effects of difference in pressure and difference in temperature may be nearly counterbalanced, with the result that the density of the air in a cyclone, as this term is used in meteorology, may be very little different from that in an anticyclone, other things being equal.

Thus far only the effect of temperature on density has been mentioned, and when once the density is known, no further reference to the temperature of the air in this connection need be made.

The second and independent influence of the temperature of the air is upon the air resistance as influenced by the elasticity, acting upon the velocity of sound. The elasticity of the air affecting the velocity of longitudinal wave motion in the air is a function of the temperature, but within reasonable bounds is unaffected by changes in density. The use of elasticity in this connection must not be confused with its occasional use as referring to "elastic pressure" or the barometric pressure in free air.

Normal temperature for a trajectory is taken as 15° C. = 59° F. = 518.4° F. (abs.), the latter being measured from absolute zero. As the projectile rises the temperature drops, a fall of over 100° F. being registered at 10 kilometers altitude. The fall in temperature refers to measurements in free air. Observations with rise in altitude but at the surface of the earth, as up the side of a mountain or cliff, show a different law, the drop being sometimes expressed as about 1° F. for each 105 meters in altitude. In free air the drop is very roughly 11° F. per kilometer. When variations in temperature are considered the resistance function $B(v) = \frac{1}{v} G(v)$ may be put in the form $B_1(v/c)$, where c is a current notation for the velocity of sound. Thus it is only the ratio of the velocity of the projectile to the velocity of sound which affects the values of B_1. The value of c itself is proportional to the square root of the absolute temperature. The change in G for a change in temperature is obtained from the function $G(v) = vB(v)$, the factor v appearing outside of B not being affected by temperature considerations; thus, with the quadratic law, there would be no correction for change in elasticity.

The Standard Temperature.

In the construction of a ready reference table for ranges, corrections for the curvature of the earth and for an assumed standard temperature structure are made. The standard temperature structure assumed was obtained for this work as follows:

The table of European international mean August temperature for air aloft, as given under the reference "Meteorology" in the Encyclopedia Britannica (11th ed.), was taken, using the first 10 kilometers only, the value at each altitude being diminished by the same constant temperature difference to obtain 15° C.

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as ground temperature. The values thus secured were compared with the mean summer temperature gradient for the United States now being published by the Weather Bureau, Washington, and for which fortunately 15° C. is the ground value. The fit was practically perfect save that near 10 kilometers above sea level the United States average drops slightly below the other values obtained. Ignoring variations from static equilibrium, changes in the value of the gravitational attraction, and the nonhomogeneity of the air, it should be possible to determine any two of the three variables—density, temperature, pressure—in terms of the third, as functions of the altitude by means of the two relations afforded by:

(1) The "gas law" (from Boyle's and Charles' laws), that the absolute temperature is proportional to the product of the volume by the pressure.

(2) The theorem of static equilibrium, that the difference in pressure at altitude y meters and at sea level is due only to the weight of the intervening layer.

The experimental density law was assumed and dynamic influences were ignored in favor of a mean static condition, while the assumed constants were determined empirically. The following simple formula was thus secured, giving good agreement with observations both for itself and for the pressure function derived from it:

$$\text{Temperature} \begin{cases} °\text{C.} = 50 - 35 \times 10^{-.000045y} \\ °\text{F.} = 122 - 63 \times 10^{-.000045y} \end{cases}$$

where y is the altitude in meters above sea level.

The above standard temperature represents fairly mean midsummer temperature for the United States. For mean midwinter temperature subtract 12.5° C. = 22.5° F. throughout.

The Standard Pressure.

The two laws mentioned give for $\delta(y)$, $t(y)$, $p(y)$, as density, absolute temperature, and pressure, respectively;

(1) $t(y) = kp(y)/\delta(y)$

(2) $p(o) - p(y) = (.0001036c') \int_o^y \delta(y)\, dy,$

where k and $.0001036c'$ are taken arbitrarily as constants of proportionality.

Since $\delta(y)$ is proportional to $e^{-.0001036y}$, and equal, say, to $(c/c')e^{-.0001036y}$, we have

$$p(y) = ce^{-.0001036y} + [p(o) - c]$$

and hence

$$t(y) = c'k + c'k[p(o)/c - 1]e^{-.0001036y}.$$

The two coefficients $c'k$ and $c'k[p(o)/c-1]$ are adjusted to the data with the result that for absolute temperature in the standard scale,

$$t(y) = (273 + 50) - 35e^{-.0001036y}$$

giving for $p(o)/c - 1$, the value $-35/323 = -.1084$. Thus

$$p(y) = c(e^{-.0001036y} - .1084)$$

whence also $p(o) = .8916c$.

For standard pressure in millimeters, $p(o) = 750.00$, hence $c = 841.18$. A discussion of the units follows.

The order followed as to the derivation and selection of empirical coefficients is that suggested by sensitiveness. Thus there has been secured a set of these functions for density, absolute temperature, and pressure of free air in terms of the altitude, all agreeing very closely with observations in this country and abroad, instead of one perhaps slightly closer by some method of determining a mean at the cost of making the other two differ hopelessly from observational data.

Pressure values for meteorological data have been given for scientific purposes in the metric system. Formerly the custom was to indicate these pressures in millimeters of mercury, as read from the mercury barometer and corrected to standard conditions. In 1908 the meteorological office of Great Britain (M. O. publication No. 202) adopted the megadyne per square centimeter as the unit of pressure. More recently in this country the "millibar" or 1/1000 megadyne (hence = 1 kilodyne/cm²) has been current. The following is quoted from the above-cited publication, "The Free Atmosphere in the Region of the British Isles," W. H. Dines:

"A megadyne per square centimeter is a million times the unit of pressure in the C. G. S. system, which is universally adopted for electrical and magnetic units. The reason for adopting this unit is that, in the examination of the results for the upper air, the actual number of inches or millimeters in the pressure at any level is of little importance compared with the fraction of the atmosphere that is above or below the level. The

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C. G. S. system affords a very easy means of showing this fraction because 1.000 megadyne per square centimeter is practically equivalent to 750 millimeters of mercury at sea level in latitude 45°, * * *. In a practical sense, therefore, a megadyne per square centimeter is "a C. G. S. atmosphere" and the fraction which gives the observed pressure in megadynes per square centimeter gives for all practical purposes the fraction of the atmosphere which remains above the point of observation. At the conference on Scientific Aeronautics held at Monaco in April of this year, Prof. Koppen proposed the adoption of the megadyne per square centimeter, to be named a "bar," for the publication of pressure values, as a modification of a proposal by Prof. McAdie, of the United States Weather Bureau, to express pressures as fractions of an atmosphere."

For ballistic purposes standard sea-level conditions have been assumed by many nations, in particular the United States and France, as involving a pressure of 750 mm. and a temperature of 15° C., at the ground. These correspond to a convenient average of summer conditions in this country only in so far as temperature is concerned, a mean estimate of pressure at sea level being frequently taken as 762.5 mm. by the meteorological service. A standard pressure of 750 mm. having been adopted, however, the standard pressure structure is taken so as to yield an initial value to conform with this usage.

The French in certain recent confidential pamphlets have proceeded along these lines and obtained functions to represent density, temperature, and pressure. The coefficients, instead of being adjusted to secure the best fit, have been derived upon theoretic grounds justifiable only under strictly static equilibrium. Two features of the French tabulations which unfavorably distinguish their results from those here given are:

First. The assumed standard conditions for ballistic investigations are not those given at any altitude by the so-called standard functions, although approximately represented at an altitude zero.

Second. The discrepancy at higher altitudes between the most favorable observed results and the tabulated values is excessive, some 20° C. in temperature being a typical discrepancy.

Like those of the French, however, the mathematical assumptions made in this paper are to be interpreted only as holding within the lower air strata. Thus the expression for the pressure can not be supposed to hold to indefinitely great heights, and integration from infinity, under the assumption that the pressure at infinity is zero, is of course out of the question, even were the gravitational field regarded as unaltered to infinity.

The standard density in kilograms at sea level is to be taken as that which corresponds in the mean, to the pressure of 750 mm. of mercury, at the temperature 15° C. and at saturation, 78 per cent. This is found to give 1.2034 kilograms per cubic meter of air, rather than the mean value at mean pressure and laboratory standard temperature.

The Curvature of Sea Level.

The mere curvature of the earth, or rather of the sea level, presents phenomena which for long ranges must be taken into consideration. For convenience, the corrections may be made separately for the ascending and for the descending branch under the reference system here adopted. The corrections are of the same type for the two branches, so that only the ascending branch will be treated explicitly.

In the figure let $GEFB$ be a section of the earth, C being the earth's center. Let G be the gun (at sea level) and GHF a trajectory, with H the highest point, and F (which will be called the "point of fall") the intersection of the trajectory with the sea level. The tangent GA, arc GE, semichord GD are different, but their lengths differ only negligibly for all artillery trajectories. Their assumed common length will be called the *distance, d*. The difference $\overline{ED}$ between the ordinate measured from a chord through the gun and when measured from the earth's surface will be called the "curvature" k, for the given distance d. The curvature k is measured in meters, d in kilometers. The angle AGD is equal to the angle GCE and is denoted by σ. Thus $\tan\sigma$ is the "slope of sea level" referred to the maximum ordinate as vertical. For our purposes either $\tan\sigma$ or $\sin\sigma$ may be taken as d/R when R is the earth's radius. Now, for small angles AE and ED are sensibly equal, and each may be called the curvature k. By plane geometry $\overline{AE}\times\overline{AB}=\overline{AG}^2$ or $\overline{AE}\times(\overline{AE}+\overline{EB})=\overline{AG}^2$, i. e., $\overline{AE}^2+\overline{AE}\times\overline{EB}=\overline{AG}^2$. Neglecting $\overline{AE}^2$ at the ranges considered, one has $\overline{AE}=\overline{AG}^2/\overline{EB}$, where $\overline{EB}$ is the earth's diameter. For $\overline{AG}=d$ given in kilometers, $\overline{AE}=k$ in meters and $\overline{EB}$ in kilometers we have

$$k=\frac{d^2}{12.74}.$$

The divisor 12.74 is taken as a mean value for two one-thousandths of the earth's radius in kilometers. The equatorial and polar radii are more exactly 6,378.2 km. and 6,356.6 km., respectively.

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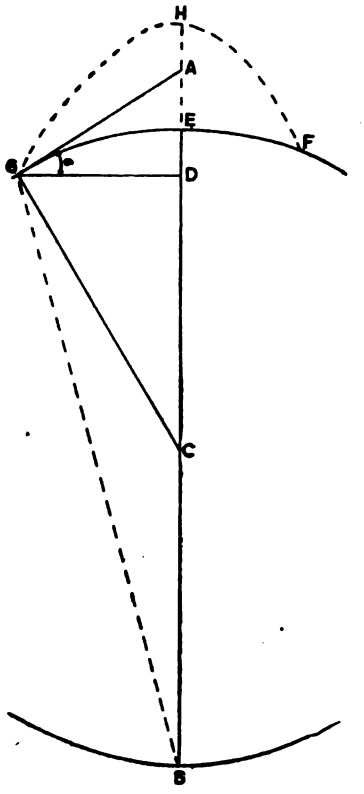

III.—Brief Historical Review.

The history of the science of ballistics is eventful and instructive. Its various periods are marked by numerous ingenious and elaborate series of experiments, and by theoretical investigations of the greatest technical excellence. Like most other sciences, the theories and methods of one period have been found inadequate for the more rigorous exigencies of later times. Occasionally this has led to an entirely new start in the mathematical methods of handling the problems.

The practical handling of artillery trajectories extends back not only to the first use of gunpowder as a propellant, but to the crossbow, the catapult, and the handsling. Previous to the early part of the sixteenth century such references as are found to the shape of the trajectory speak of the first or "violent" part as a straight line, the middle part as the arc of a circle, and the last or "natural" part as again straight.

The science of ballistics may be regarded as having been initiated by the physicist Nicholas Tartaglia, who in 1537 published a treatise on the flight of projectiles. He showed that the trajectory is curved throughout, attaining a maximum range at an angle of departure of 45°.

Galileo in his fourth Dialogue on Mechanics demonstrated that the path described by a projectile being the result of a combination of a uniform transverse motion with a uniformly accelerated vertical motion must, apart from the resistance of the air, be a parabola with axis vertical.

Newton was the first to take into account quantitatively the resistance of the air. As a deduction from fundamental physical laws, he showed that when the velocity is so little in comparison with the mobility of the air as to make it possible to ignore the diminution of pressure in the rear of a projectile, and the stream line motion of the air generally, then the resistance must vary as the square of the velocity. To test these fundamental principles and to determine the constants involved, he let fall from the dome of St. Paul's Cathedral spheres of various weights and diameters filled respectively with air, water, or mercury, thus repeating Galileo's investigation with new refinements.

Instruments to measure air resistance, in the form of rotating vanes somewhat similar to the anemometers employed by the Weather Bureau, have been used as the basis of computation by numerous investigators, such as Borda, Thibault, Hutton, Piobert, Morin, Didion, etc. Newton's results, while in themselves satisfactory, were soon found not to be directly applicable to most artillery fire, having been secured under ideal conditions not in practice realized.

Benjamin Robins published in 1742 his New Principles of Gunnery, based upon results obtained from using his "ballistic pendulum." This was the first experimental device to record with any degree of accu-

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racy the velocities of projectiles as obtained in artillery fire. While Newton's results were confirmed at velocities under about 900 f/s, Robins was able to measure velocities up to 1,700 f/s, at which the discrepancies between observation and the "square law" became enormous. Further extensive measurements by the ballistic pendulum were completed by Hutton in England in 1791 and by the Commission de Metz in France (Piobert, Morin, Didion) in 1839–40.

The practical use of the ballistic pendulum became obsolete when in 1840 Wheatstone suggested measuring the velocity of the projectile through time intervals recorded by means of electric contacts effected as the projectile passes through successive screens, at carefully measured distances.

The work of the Commission de Metz of 1856–57 and of the Commission de Gâvre 1856–61, 1873, and later to date; Bashforth's investigations, 1865–1870 and 1878–1880; Mayevski's experiments, St. Petersburg, 1869; the Krupp experiments at Meppen, 1875–1881, as analyzed by Mayevski, 1883, and extended by Zaboudski, and Hojel's firings in Holland, 1884, are the best known experimental programs previous to the present great war and are all based on screen firings.

In the course of these experiments successive modifications have arisen in the assumed form of the resistance law.

For very low velocities such as those of a pendulum, street car, etc., the stream effect of the air appears to be such that the resistance is closely proportional to the first power of the velocity. In the case of drop bombs the gradual stabilizing during flight has the effect of making the linear law more accurate than the quadratic.

Up to about 30 m/s the quadratic law with constant coefficient is still used as the most satisfactory. Hélie (1870) separated the velocities into 22 zones, with a quadratic law in each, the coefficient alone changing with the zone.

Bashforth proposed and used a cubic law, although he found it necessary to use different coefficients for different velocity intervals, inasmuch as a single algebraic formulation proved insufficient.

For all resistance laws of the form $a+cv^n$, where a, c, n are constant and v is the velocity, explicit solution is possible in finite form by means of functions of one variable, although rather cumbersome in most cases. The case where a is zero was proved in 1719 by John Bernoulli, the Swiss mathematician, and the more general case by D'Alembert in 1744. These solutions are also applicable when the velocity interval is divided into zones, with distinct values of a, c, and n in the separate zones.

J. Didion and St. Robert, on the basis of the results of the Metz tests of 1839–40, gave two-term resistance laws. Mayevski also used two-term laws, with zones as well, in some of his earlier publications. More recent tabulations which were generally adopted divided the range of velocities into zones with a law of the form cv^n in each zone, c and n changing from zone to zone. Such were given by Mayevski, 1881; Chapel, 1874; Vallier, 1894; Scheve, 1907. Still more recently Siacci, 1896, and the Gâvre Commission, have given single analytic forms available over the whole range of velocities considered.

The mathematical form of the air resistance law has gone through the following well-defined stages:

First. Resistance ignored.

Second. "Newton's law," based on theory and experiment giving a resistance law cv^2, available for low velocities.

Third. Successive simple empirical laws of one or two term form intended to be available for all velocities.

Fourth. Laws for separate zones, these laws being of the form cv^n, with c and n constant in any zone but changing from zone to zone.

Fifth. Explicit use of a tabulated function expressed indeed by a single analytic equation of as complicated form as may be necessary, but obtained so as to pass acceptably close to all the data secured from extensive observations. A single form is chosen so as to avoid computational and theoretical irregularities at the junction of zones. In the tabulated form as used this amounts merely to an extremely large number of zones, with particularly smooth junctions.

Sixth. A formulation of the resistance function as explicitly dependent not only upon velocity but upon other features variable along the trajectory, in particular decrease of air density with altitude, and more recently, change of elasticity caused by decrease of temperature with altitude.

So much for the physical side of the problem of air resistance. The mathematical methods used for solving the differential equations of the motion of the projectile show analogous periods:

First. Air resistance ignored.

Second. Complete mathematical solutions obtained by Bernoulli's method for monomial resistance laws either through the whole velocity interval or by zones.

Third. Methods of approximate solution of the exact equations, numerical integration, graphical integration, and power series methods.

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Fourth. Replacing the exact equations by mean value expressions which may be exactly integrated in simple form.

Fifth. Return to the methods of the third period.

A few words may be said about the fourth period which is associated with the name of Siacci. His methods were universally in use among ballisticians of all countries prior to the World War, and have been since the advent of the war almost universally discarded.

The exact differential equation in v and θ, neglecting other variations, is

$$g \cdot d(v \cos \theta) = v \cdot c \cdot F(v) d\theta,$$

and may be written,

$$\frac{d\theta}{\cos^2\theta} = \frac{g dx'}{x' c F\left(\frac{x'}{\cos\theta}\right) \cdot \cos\theta},$$

where $x' = v \cos \theta$. The approximation used is to replace the expression $\cos\theta$ appearing twice in the right-hand member by mean constant values σ and γ, respectively, giving as the approximate equation

$$\frac{d\theta}{\cos^2\theta} = \frac{g dx'}{\gamma \cdot x' c F\left(\frac{x'}{\sigma}\right)},$$

Now, while σ and γ are constant on a trajectory, their values may be functions of the initial conditions, so that in place of x'/σ, one writes $u = x'/\sigma$, hence

$$\frac{d\theta}{\cos^2\theta} = \frac{g}{c\gamma} \cdot \frac{du}{uF(u)}.$$

The variables θ and u are separated and the equation is integrable. The various usages in the several countries differ only in the choice of the mean values σ and γ as functions of the initial conditions, and in details of minor interest.

The fifth period was ushered in by the World War, when it was found that any such mean value methods were inadequate on account of (1) the large angles of departure to be used and accuracy required, and hence relatively great variations of $\cos\theta$ from any constant value; (2) the inadequacy of supposing that the variations in air density with altitude could be ignored in the fundamental equations; (3) the necessity of an accurate method of handling differential corrections arising from variations from range table conditions. While a reversion to older methods, the technique now employed shows of course improvements due to modern mathematical discoveries and computational experience. Methods of the same general type were developed more or less independently in all of the principal countries and at about the same time, although the explicit variables chosen as independent differed with the place. The methods introduced in this country are due chiefly to Maj. F. R. Moulton, professor of astronomy at the University of Chicago, and comparison with methods that had been just instituted abroad show these here used superior in simplicity and general convenience.

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